A HISTORY OF THE PRIME NUMBER THEOREM

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The sequence of prime numbers, which begins

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \cdots,$$

has held untold fascination for mathematicians, both professionals and amateurs alike. The basic theorem which we shall discuss in this lecture is known as the **prime number theorem** and allows one to predict, at least in gross terms, the way in which the primes are distributed. Let x be a positive real number, and let $\pi(x) =$ the number of primes $\leqq x$. Then the prime number theorem asserts that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1, \tag{1}$$

where $\log x$ denotes the natural log of x. In other words, the prime number theorem asserts that

$$\pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right), \qquad (x \to \infty), \tag{2}$$

where $o(x/\log x)$ stands for a function $f(x)$ with the property

$$\lim_{x\to\infty} \frac{f(x)}{x/\log x} = 0.$$

Actually, for reasons which will become clear later, it is much better to replace (1) and (2) by the following equivalent assertion:

$$\pi(x) = \int_2^x \frac{dy}{\log y} + o\left(\frac{x}{\log x}\right). \tag{3}$$

To prove that (2) and (3) are equivalent, it suffices to integrate

$$\int_2^x \frac{dy}{\log y}$$

once by parts to get

$$\int_2^x \frac{dy}{\log y} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dy}{\log^2 y}. \tag{4}$$

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However, for $x \geqq 4$,

$$\int_2^x \frac{dy}{\log^2 y} = \int_2^{\sqrt{x}} \frac{dy}{\log^2 y} + \int_{\sqrt{x}}^x \frac{dy}{\log^2 y}$$

$$\leqq \sqrt{x} \cdot \frac{1}{\log^2 2} + x \cdot \frac{1}{\log^2(\sqrt{x})} \tag{5}$$

$$= o\left(\frac{x}{\log x}\right),$$

where we have used the fact that $1/\log^2 x$ is monotone decreasing for $x > 1$. It is clear that (4) and (5) show that (2) and (3) are equivalent to one another. The advantage of the version (3) is that the function

$$\mathrm{Li}(x) = \int_2^x \frac{dy}{\log x},$$

called the **logarithmic integral**, provides a much closer numerical approximation to $\pi(x)$ than does $x/\log x$. This is a rather deep fact and we shall return to it.

In this lecture, I should like to explore the history of the ideas which led up to the prime number theorem and to its proof, which was not supplied until some 100 years after the first conjecture was made. The history of the prime number theorem provides a beautiful example of the way in which great ideas develop and interrelate, feeding upon one another ultimately to yield a coherent theory which rather completely explains observed phenomena.

The very conception of a prime number goes back to antiquity, although it is not possible to say precisely when the concept first was clearly formulated. However, a number of elementary facts concerning the primes were known to the Greeks. Let us cite three examples, all of which appear in Euclid:

(i) (*Fundamental Theorem of Arithmetic*): Every positive integer n can be written as a product of primes. Moreover, this expression of n is unique up to a rearrangement of the factors.

(ii) There exist infinitely many primes.

(iii) The primes may be effectively listed using the so-called "sieve of Eratosthenes".

We will not comment on (i), (iii) any further, since they are part of the curriculum of most undergraduate courses in number theory, and hence are probably familiar to most of you. However, there is a proof of (ii) which is quite different from Euclid's well-known proof and which is very significant to the history of the prime number theorem. This proof is due to the Swiss mathematician Leonhard Euler and dates from the middle of the 18th century. It runs as follows:

Assume that $p_1, \cdots, p_N$ is a complete list of all primes, and consider the product

(6) $$\prod_{i=1}^{N}\left(1-\frac{1}{p_i}\right)^{-1}=\prod_{i=1}^{N}\left(1+\frac{1}{p_i}+\frac{1}{p_i^2}+\cdots\right).$$

Since every positive integer n can be written uniquely as a product of prime powers, every unit fraction $1/n$ appears in the formal expansion of the product (6). For example, if $n = p_1^{a_1}\cdots p_N^{a_N}$, then $1/n$ occurs from multiplying the terms

$$1/p_1^{a_1},\ 1/p_2^{a_2},\cdots,\ 1/p_N^{a_N}.$$

Therefore, if R is any positive integer,

(7) $$\prod_{i=1}^{N}\left(1-\frac{1}{p_i}\right)^{-1}\geqq\sum_{n=1}^{R}1/n.$$

However, as $R \to \infty$, the sum on the right hand side of (7) tends to infinity, which contradicts (7). Thus, $p_1,\cdots,p_N$ cannot be a complete list of all primes. We should make two comments about Euler's proof: First, it links the Fundamental Theorem of Arithmetic with the infinitude of primes. Second, it uses an analytic fact, namely the divergence of the harmonic series, to conclude an arithmetic result. It is this latter feature which became the cornerstone upon which much of 19th century number theory was erected.

The first published statement which came close to the prime number theorem was due to Legendre in 1798 [**8**]. He asserted that $\pi(x)$ is of the form $x/(A\log x + B)$ for constants A and B. On the basis of numerical work, Legendre refined his conjecture in 1808 [**9**] by asserting that

$$\pi(x) = \frac{x}{\log x + A(x)},$$

where $A(x)$ is "approximately 1.08366 $\cdots$". Presumably, by this latter statement, Legendre meant that

$$\lim_{x\to\infty} A(x) = 1.08366.$$

It is precisely in regard to $A(x)$, where Legendre was in error, as we shall see below. In his memoir [**9**] of 1808, Legendre formulated another famous conjecture. Let k and l be integers which are relatively prime to one another. Then Legendre asserted that there exist infinitely many primes of the form $l + kn$ $(n = 0, 1, 2, 3, \cdots)$. In other words, if $\pi_{k,l}(x)$ denotes the number of primes p of the form $l + kn$ for which $p \leqq x$, then Legendre conjectured that

(8) $$\pi_{k,l}(x)\to\infty \quad \text{as} \quad x\to\infty.$$

Actually, the proof of (8) by Dirichlet in 1837 [**2**] provided several crucial ideas on how to approach the prime number theorem.

Although Legendre was the first person to publish a conjectural form of the prime number theorem, Gauss had already done extensive work on the theory of primes in 1792–3. Evidently Gauss considered the tabulation of primes as some sort of pastime and amused himself by compiling extensive tables on how the primes distribute themselves in various intervals of length 1000. We have included some of Gauss' tabulations as an Appendix. The first table, excerpted from [3, p. 436], covers the primes from 1 to 50,000. Each entry in the table represents an interval of length 1000. Thus, for example, there are 168 primes from 1 to 1000; 135 from 1001 to 2000; 127 from 3001 to 4000; and so forth. Gauss suspected that the density with which primes occured in the neighborhood of the integer n was $1/\log n$, so that the number of primes in the interval $[a, b)$ should be approximately equal to

$$\int_a^b \frac{dx}{\log x}.$$

In the second set of tables, samples from [4, pp. 442–3], Gauss investigates the distribution of primes up to 3,000,000 and compares the number of primes found with the above integral. The agreement is striking. For example, between 2,600,000 and 2,700,000, Gauss found 6762 primes, whereas

$$\int_{2,600,000}^{2,700,000} \frac{dx}{\log x} = 6761.332.$$

Gauss never published his investigations on the distribution of primes. Nevertheless, there is little reason to doubt Gauss' claim that he first undertook his work in 1792–93, well before the memoir of Legendre was written. Indeed, there are several other known examples of results of the first rank which Gauss proved, but never communicated to anyone until years after the original work had been done. This was the case, for example, with the elliptic functions, where Gauss preceded Jacobi, and with Riemannian geometry, where Gauss anticipated Riemann. The only information beyond Gauss' tables concerning Gauss' work in the distribution of primes is contained in an 1849 letter to the astronomer Encke. We have included a translation of Gauss' letter.

In his letter Gauss describes his numerical experiments and his conjecture concerning $\pi(x)$. There are a number of remarkable features of Gauss' letter. On the second page of the letter, Gauss compares his approximation to $\pi(x)$, namely $\mathrm{Li}(x)$, with Legendre's formula. The results are tabulated at the top of the second page and Gauss' formula yields a much larger numerical error. In a very prescient statement, Gáuss defends his formula by noting that although Legendre's formula yields a smaller error, the rate of increase of Legendre's error term is much greater than his own. We shall see below that Gauss anticipated what is known as the "Riemann hypothesis." Another feature of Gauss' letter is that he casts doubt on Legendre's assertion about $A(x)$. He asserts that the numerical evidence does not support any conjecture about the limiting value of $A(x)$.

Gauss' calculations are awesome to contemplate, since they were done long before the days of high-speed computers. Gauss' persistence is most impressive. However, Gauss' tables are not error-free. My student, Edward Korn, has checked Gauss' tables using an electronic computer and has found a number of errors. We include the corrected entries in an appendix. In spite of these (remarkably few) errors, Gauss' calculations still provide overwhelming evidence in favor of the prime number theorem. Modern students of mathematics should take note of the great care with which data was compiled by such giants as Gauss. Conjectures in those days were rarely idle guesses. They were usually supported by piles of laboriously gathered evidence.

The next step toward a proof of the prime number theorem was a step in a completely different direction, and was taken by Dirichlet in 1837 [**2**]. In a beautiful memoir, Dirichlet proved Legendre's conjecture (8) concerning the infinitude of primes in an arithmetic progression. Dirichlet's work contained two radically new ideas, which we should discuss in some detail.

Let $\mathbb{Z}_n$ denote the ring of residue classes modulo n, and let $\mathbb{Z}_n^\times$ denote the group of units of $\mathbb{Z}_n$. Then $\mathbb{Z}_n^\times$ is the so-called "group of reduced residue classes modulo n" and consists of those residue classes containing an element relatively prime to n. If k is an integer, let us denote by $\bar{k}$ its residue class modulo n. Dirichlet's first brilliant idea was to introduce the **characters** of the group $\mathbb{Z}_n^\times$; that is, the homomorphisms of $\mathbb{Z}_n^\times$ into the multiplicative group $\mathbb{C}^\times$ of non-zero complex numbers. If χ is such a character, then we may associate with χ a function (also denoted χ) from the semi-group $\mathbb{Z}^*$ of non-zero integers as follows: Set

$$\chi(a) = \chi(\bar{a}) \text{ if } (a,n) = 1$$
$$0 \text{ otherwise.}$$

Then it is clear that $\chi\colon \mathbb{Z}^* \to \mathbb{C}^\times$ and has the following properties:

$$\text{(i)} \quad \chi(a+n) = \chi(a),$$
$$\text{(ii)} \quad \chi(aa') = \chi(a)\chi(a'),$$
$$\text{(iii)} \quad \chi(a) = 0 \text{ if } (a,n) \neq 1,$$
$$\text{(iv)} \quad \chi(1) = 1.$$

A function $\chi\colon \mathbb{Z}^* \to \mathbb{C}^\times$ satisfying (i)–(iv) is called a **numerical character** modulo n. Dirichlet's main result about such numerical characters was the so-called **orthogonality relations**, which assert the following:

(A) $$\sum_a \chi(a) = \phi(n) \text{ if } \chi \text{ is identically 1,}$$
$$0 \quad \text{otherwise,}$$

where a runs over a complete system of residues modulo n;

(B) $$\sum_{\chi} \chi(a) = \phi(n) \text{ if } a \equiv 1 \pmod{n}, \quad 0 \quad \text{otherwise},$$

where χ runs over all numerical characters modulo n. Dirichlet's ideas gave birth to the modern theory of duality on locally compact abelian groups.

Dirichlet's second great idea was to associate to each numerical character modulo n and each real number $s > 1$, the following infinite series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{9}$$

It is clear that the series converges absolutely and represents a continuous function for $s > 1$. However, a more delicate analysis shows that the series (9) converges (although not absolutely) for $s > 0$ and represents a continuous function of s in this semi-infinite interval *provided that χ is not identically* 1. The function $L(s, \chi)$ has come to be called a **Dirichlet L-function.**

Note the following facts about $L(s, \chi)$: First $L(s, \chi)$ has a product formula of the form

$$L(s, \chi) = \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \qquad (s > 1), \tag{10}$$

where the product is taken over all primes p. The proof of (10) is very similar to the argument given above in Euler's proof of the infinity of prime numbers. Therefore, by (10),

$$\begin{aligned} \log L(s, \chi) &= -\sum_{p} \log\left(1 - \frac{\chi(p)}{p^s}\right) \\ &= -\sum_{p} \sum_{m=1}^{\infty} \frac{\chi(p^m)}{mp^{ms}}. \end{aligned} \tag{11}$$

Dirichlet's idea in proving the infinitude of primes in the arithmetic progression a, $a + n$, $a + 2n, \cdots, (a, n) = 1$, was to imitate, somehow, Euler's proof of the infinitude of primes, by studying the function $L(s, \chi)$ for s near 1. The basic quantity to consider is

$$\begin{aligned} \sum_{\chi} \chi(a)^{-1} \log L(s, \chi) &= -\sum_{p} \sum_{m=1}^{\infty} \sum_{\chi} \frac{\chi(a)^{-1}\chi(p^m)}{mp^{ms}} \\ &= -\sum_{p} \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \chi(a)^{-1}\chi(p^m), \end{aligned} \tag{12}$$

where we have used (11). Let a^* be an integer such that $aa^* \equiv 1 \pmod{n}$. Then $\chi(a^*) = \chi(a)^{-1}$ by (i)–(iv). Moreover,

$$\sum_{\chi} \chi(a)^{-1}\chi(p^m) = \sum_{\chi} \chi(a^*p^m)$$

$$(13) \qquad = \phi(n) \text{ if } a^*p^m \equiv 1 \pmod{n}$$

$$0 \text{ otherwise.}$$

However, $a^*p^m \equiv 1 \pmod{n}$ is equivalent to $p^m \equiv a \pmod{n}$. Therefore, by (12) and (13), we have

$$(14) \qquad \sum_{\chi} \chi(a)^{-1} \log L(s,\chi) = -\phi(n) \sum_{\substack{p \\ p^m \equiv a (\bmod n)}} \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}.$$

Thus, finally, we have

$$(15) \qquad -\frac{1}{\phi(n)} \sum_{\chi} \chi(a) \log L(s,\chi) - \sum_{\substack{p \\ p^m \equiv a (\bmod n)}} \sum_{m=2}^{\infty} \frac{1}{mp^{mp}} = \sum_{\substack{p \\ p \equiv a (\bmod n)}} \frac{1}{p^s} \qquad (s > 1).$$

From (15), we immediately see that in order to prove that there are infinitely many primes $p \equiv a \pmod{n}$, it is enough to show that the function

$$\sum_{p \equiv a (\bmod n)} \frac{1}{p^s}$$

tends to $+\infty$ as s approaches 1 from the right. But it is fairly easy to see that as $s \to 1+$, the sum

$$\sum_{\substack{p \\ p^m \equiv a (\bmod n)}} \sum_{m=2}^{\infty} \frac{1}{mp^{ms}}$$

remains bounded. Thus, it suffices to show that

$$-\frac{1}{\phi(n)} \sum_{\chi} \chi(a)^{-1} \log L(s,\chi) \to +\infty \qquad (s \to 1+).$$

However, if χ_0 denotes the character which is identically 1, then it is easy to see that

$$-\frac{1}{\phi(n)} \chi_0(a)^{-1} L(s,\chi_0) \to +\infty \text{ as } s \to 1+.$$

Therefore, it is enough to show that if $\chi \neq \chi_0$, then log $L(s,\chi)$ remains bounded as $s \to 1+$. We have already mentioned that $L(s,\chi)$ is continuous for $s > 0$ if $\chi \neq \chi_0$. Therefore, it suffices to show that $L(1,\chi) \neq 0$. And this is precisely what Dirichlet showed.

Dirichlet's theorem on primes in arithmetic progressions was one of the major achievements of 19th century mathematics, because it introduced a fertile new idea into number theory—that analytic methods (in this case the study of the Dirichlet L-series) could be fruitfully applied to arithmetic problems (in this case the problem of primes in arithmetic progressions). To the novice, such an application of analysis to number theory would seem to be a waste of time. After all, number theory is the study of the discrete, whereas analysis is the study of the continuous; and what should one have to do with the other! However, Dirichlet's 1837 paper was the beginning of a revolution in number-theoretic thought, the substance of which was to apply analysis to number theory. At first, undoubtedly, mathematicians were very uncomfortable with Dirichlet's ideas. They regarded them as very clever devices, which would eventually be supplanted by completely arithmetic ideas. For although analysis might be useful in proving results about the integers, surely the analytic tools were not intrinsic. Rather, they entered the theory of the integers in an inessential way and could be eliminated by the use of suitably sophisticated arithmetic. However, the history of number theory in the 19th century shows that this idea was eventually repudiated and the rightful connection between analysis and number theory came to be recognized.

The first major progress toward a proof of the prime number theorem after Dirichlet was due to the Russian mathematician Tchebycheff in two memoirs [**12, 13**] written in 1851 and 1852. Tchebycheff introduced the following two functions of a real variable x:

$$\theta(x) = \sum_{p \leqq x} \log p,$$

$$\psi(x) = \sum_{p^m \leqq x} \log p,$$

where p runs over primes and m over positive integers. Tchebycheff proved that the prime number theorem (1) is equivalent to either of the two statements

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1, \tag{16}$$

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = 1. \tag{17}$$

Moreover, Tchebycheff proved that if $\lim_{x \to \infty} (\theta(x)/x)$ exists, then its value must be 1. Furthermore, Tchebycheff proved that

$$.92129 \leqq \liminf_{x \to \infty} \frac{\pi(x)}{x/\log x} \leqq 1 \leqq \limsup_{x \to \infty} \frac{\pi(x)}{x/\log x} \leqq 1.10555. \tag{18}$$

Tchebycheff's methods were of an elementary, combinatorial nature, and as such were not powerful enough to prove the prime number theorem.

The first giant strides toward a proof of the prime number theory were taken by B. Riemann in a memoir [**10**] written in 1860. Riemann followed Dirichlet in connecting problems of an arithmetic nature with the properties of a function of a continuous variable. However, where Dirichlet considered the functions $L(s,\chi)$ as functions of a real variable s, Riemann took the decisive step in connecting arithmetic with the theory of functions of a complex variable. Riemann introduced the following function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{19}$$

which has come to be known as the **Riemann zeta function**. It is reasonably easy to see that the series (19) converges absolutely and uniformly for s in a compact subset of the half-plane $\mathrm{Re}(s) > 1$. Thus, $\zeta(s)$ is analytic for $\mathrm{Re}(s) > 1$. Moreover, by using the same sort of argument used in Euler's proof of the infinitude of primes, it is easy to prove that

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\mathrm{Re}\,(s) > 1), \tag{20}$$

where the product is extended over all primes p. Euler's proof of the infinitude of primes suggests that the behavior of $\zeta(s)$ for $s = 1$ is somehow connected with the distribution of primes. And, indeed, this is the case.

Riemann proved that $\zeta(s)$ can be analytically continued to a function which is meromorphic in the whole s-plane. The only singularity of $\zeta(s)$ occurs at $s = 1$ and the Laurent series about $s = 1$ looks like

$$\zeta(s) = \frac{1}{s-1} + a_0 + a_1(s-1) + \cdots. \tag{21}$$

Moreover, if we set

$$R(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \tag{22}$$

then $R(s)$ is an entire function of s and satisfies the functional equation

$$R(s) = R(1-s). \tag{23}$$

To see the immediate connection between the Riemann zeta function and the distribution of primes, let us return to Euler's proof of the infinitude of primes. A variation on the idea of Euler's proof is as follows: Suppose that there were only finitely many primes $p_1, \cdots, p_N$. Then by (20), $\zeta(s)$ would be bounded as s tends to 1, which contradicts equation (21). Thus, the presence of a pole of $\zeta(s)$ at $s = 1$ immediately implies that there are infinitely many primes. But the connection between the zeta function and the distribution of primes runs even deeper.

Let us consider the following heuristic argument: From equation (20), it is easy to deduce that

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{m=1}^{\infty} (\log p) p^{-ms} \qquad (\operatorname{Re}(s) > 1). \tag{24}$$

Moreover, by residue calculus, it is easy to verify that

$$\lim_{T\to\infty} \frac{1}{2m} \int_{2-iT}^{2+iT} \frac{a^s}{s} ds = \begin{matrix} 1, & x < 1 \\ 0, & x > 1. \end{matrix} \tag{25}$$

Therefore, assuming that interchange of limit and summation is justified, we see that for x not equal to an integer, we have

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{2m} \int_{2-iT}^{2+iT} \frac{x^s}{s} \frac{\zeta'(s)}{\zeta(s)} ds &= \sum_p \sum_{m=1}^{\infty} (\log p) \lim_{T\to\infty} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \left(\frac{x}{p^m}\right)^s \frac{1}{s} ds \\ &= \sum_{p^m \leqq x} \log p \text{ (by equation (25))} \\ &= \psi(x). \end{aligned} \tag{26}$$

Thus, we see that there is an intimate connection between the function $\psi(x)$ and $\zeta(s)$. This connection was first exploited by Riemann, in his 1860 paper.

Note that the function

$$\frac{x^s}{s} \frac{\zeta'(s)}{\zeta(s)} \tag{27}$$

has poles at $s = 0$ and at all zeroes ρ of $\zeta(s)$. Moreover, note that by equation (20), we see that $\zeta(s) \neq 0$ for $\operatorname{Re}(s) > 1$. Therefore, all zeroes of $\zeta(s)$ lie in the half-plane $\operatorname{Re}(s) \leqq 1$. Further, since $R(s)$ is entire and $\zeta(s) \neq 0$ for $\operatorname{Re}(s) > 1$, the functional equation (23) implies that the only zeroes of $\zeta(s)$ for which $\operatorname{Re}(s) < 0$ are at $s = -2, -4, -6, -8, \cdots$, and these are all simple zeroes and are called the **trivial zeroes** of $\zeta(s)$. Thus, we have shown that all non-trivial zeroes of $\zeta(s)$ lie in the strip $0 \leqq \operatorname{Re}(s) \leqq 1$. This strip is called the **critical strip**. The residue of (27) at a non-trivial zero ρ is

$$\frac{x^p}{\rho}.$$

Thus, if σ is a large negative number, and if $C_{\sigma,T}$ denotes the rectangle with vertices $\sigma \pm iT$, $2 \pm iT$, then Cauchy's theorem implies that

$$\frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{x^s}{s} \frac{\zeta'(s)}{\zeta(s)} ds = \frac{1}{2\pi i} \left[\int_{\sigma-iT}^{\sigma+iT} + \int_{\sigma+iT}^{2+iT} + \int_{2-iT}^{\sigma-iT} \right] \frac{x^s}{s} \frac{\zeta'(s)}{\zeta(s)} ds + R(\sigma, T), \tag{28}$$

where $R(\sigma, T)$ denotes the sum of the residues of the function (27) at the poles inside

$C_{\sigma,T}$. By letting σ and T tend to infinity and by applying equations (26) and (28), Riemann arrived at the following remarkable formula, known as **Riemann's explicit formula**

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}), \tag{29}$$

where ρ runs over all non-trivial zeroes of the Riemann zeta function. Riemann's formula is surprising for at least two reasons. First, it connects the function $\psi(x)$, which is connected with the distribution of primes, with the distribution of the zeroes of the Riemann zeta function. That there should be any connection at all is amazing. But, secondly, the formula (29) explicitly puts in evidence a form of the prime number theorem by equating $\psi(x)$ with x plus an error term which depends on the zeroes of the zeta function. If we denote this error term by $E(x)$, then we see that the prime number theorem is equivalent to the assertion

$$\lim_{x\to\infty} \frac{E(x)}{x} = 0, \tag{30}$$

which, in turn, is equivalent to the assertion

$$\lim_{x\to\infty} \frac{1}{x} \sum_{\rho} \frac{x^{\rho}}{\rho} = 0. \tag{31}$$

Riemann was unable to prove (31), but he made a number of conjectures concerning the distributions of the zeroes ρ from which the statement (31) follows immediately. The most famous of Riemann's conjectures is the so-called **Riemann hypothesis,** which asserts that all non-trivial zeroes of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$, which is the line of symmetry of the functional equation (23). This conjecture has resisted all attempts to prove it for more than a century and is one of the most celebrated open problems in all of mathematics. However, if the Riemann hypothesis is true, then

$$\left|\frac{x^{\rho}}{\rho}\right| = x^{\frac{1}{2}} \frac{1}{|\rho|}$$

and from this fact and equation (29), it is possible to prove that

$$\psi(x) = x + O(x^{\frac{1}{2}+\varepsilon}) \tag{32}$$

for every $\varepsilon > 0$, where $O(x^{\frac{1}{2}+\varepsilon})$ denotes a function $f(x)$ such that $f(x)/x^{\frac{1}{2}+\varepsilon}$ is bounded for all large x. Thus, the Riemann hypothesis implies (31) in a trivial way, and hence the prime number theorem follows from the Riemann hypothesis. What is perhaps more striking is the fact that *if* (32) *holds then the Riemann hypothesis is true.* Thus, the prime number theorem in the sharp form (32) is equivalent to the Riemann hypothesis. We see, therefore, that the connection between the zeta function and the distribution of primes is no accidental affair, but somehow is woven into the fabric of nature.

In his memoir, Riemann made many other conjectures. For example, if $N(T)$ denotes the number of non-trivial zeroes ρ of $\zeta(s)$ such that $-T \leqq \operatorname{Im}(\rho) \leqq T$, then Riemann conjectured that

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log(2\pi)}{2\pi} T + O(\log T). \tag{33}$$

The formula (33) was first proven by von-Mangoldt in 1895 [**14**]. An interesting line of research has been involved in obtaining estimates for the number of non-trivial zeroes ρ on the line $\operatorname{Re}(s) = \frac{1}{2}$. Let $M(T)$ denote the number of ρ such that $\operatorname{Re}(s) = \frac{1}{2}, -T \leqq \operatorname{Im}(s) \leqq T$. Then Hardy [**6**] in 1912, proved that $M(T)$ tends to infinity as T tends to infinity. Later, Hardy [**7**] improved his argument to prove that $M(T) > AT$, where A is a positive constant, not depending on T. The ultimate result of this sort was obtained by Atle Selberg in 1943 [**11**]. He proved that $M(T) > AT \log T$ for some positive constant A. In view of equation (33), Selberg's result shows that a positive percentage of the zeroes of $\zeta(s)$ actually lie on the line $\operatorname{Re}(s) = \frac{1}{2}$. This result represents the best progress made to date in attempting to prove the Riemann hypothesis.

Fortunately, it is not necessary to prove the Riemann hypothesis in order to prove the prime number theorem in the form (17). However, it is necessary to obtain some information about the distribution of the zeroes of $\zeta(s)$. Such information was obtained independently by Hadamard [**5**] and de la Vallee Poussin [**1**] in 1896, thereby providing the first complete proofs of the prime number theorem. Although their proofs differ in detail, they both establish the existence of a zero-free region for $\zeta(s)$, the existence of which serves as a substitute for the Riemann hypotheses in the reasoning presented above. More specifically, they proved that there exist constants a, t_0 such that $\zeta(\sigma + it) \neq 0$ if $\sigma \geqq 1 - 1/a \log |t|, |t| \geqq t_0$. This zero-free region allows one to prove the prime number theorem in the form

$$\psi(x) = x + O(xe^{-c(\log x)^{1/14}}). \tag{34}$$

Please note, however, that the error term in (34) is much larger than the error term predicted by the Riemann hypothesis.

Thus, the prime number theorem was finally proved after a century of hard work by many of the world's best mathematicians. It is grossly unfair to attribute proof of such a theorem to the genius of a single individual. For, as we have seen, each step in the direction of a proof was conditioned historically by the work of preceding generations. On the other hand, to deny that there is genius in the work which led up to the ultimate proof would be equally unfair. For at each step in the chain of discovery, brilliant and fertile ideas were discovered, and provided the material out of which to fashion the next link.

APPENDIX A: **Samples from Gauss' Tables.** TABLE 1 (*Werke*, *II*, p. 436)

1	168	26	98
2	135	27	101
3	127	28	94
4	120	29	98
5	119	30	92
6	114	31	95
7	117	32	92
8	107	33	106
9	110	34	100
10	112	35	94
11	106	36	92
12	103	37	99
13	109	38	94
14	105	39	90
15	102	40	96
16	108	41	88
17	98	42	101
18	104	43	102
19	94	44	85
20	102	45	96
21	98	46	86
22	104	47	90
23	100	48	95
24	104	49	89
25	94	50	98

The frequency of primes. TABLE 2 (*Werke*, *II*, p. 443) 2000000···3000000

	210	220	230	240	250	260	270	280	290	300	
0							1				1
1	3	2	2	4	1	3	4	2	2	2	25
2	10	9	9	11	9	5	10	7	15	13	98
3	32	27	29	32	37	35	28	43	30	44	337
4	69	69	73	86	78	88	71	95	85	64	778
5	119	146	138	136	147	136	158	135	140	153	1408
6	197	183	179	176	193	194	195	195	179	187	1878
7	204	201	205	194	189	180	201	188	222	214	1998
8	157	168	168	158	151	170	142	145	132	134	1525
9	115	109	113	112	102	88	96	87	109	103	1034
10	63	52	44	55	58	58	53	67	53	58	561
11	21	18	30	28	23	24	22	24	18	15	223
12	8	9	10	7	7	13	17	9	8	11	99
13	2	4		1	5	6	1	2	5	1	27
14		3					1		2		6
15										1	1
16											
17								1			1
	6874	6857	6849	6787	6766	6804	6762	6714	6744	6705	6862

Appendix B: Gauss' Letter to Enke.

My distinguished friend:

Your remarks concerning the frequency of primes were of interest to me in more ways than one. You have reminded me of my own endeavors in this field which began in the very distant past, in 1792 or 1793, after I had acquired the Lambert supplements to the logarithmic tables. Even before I had begun my more detailed investigations into higher arithmetic, one of my first projects was to turn my attention to the decreasing frequency of primes, to which end I counted the primes in several chiliads (*intervals of length* 1000; *Trans.*) and recorded the results on the attached white pages. I soon recognized that behind all of its fluctuations, this frequency is on the average inversely proportional to the logarithm, so that the number of primes below a given bound n is approximately equal to

$$\int \frac{dn}{\log n},$$

where the logarithm is understood to be hyperbolic. Later on, when I became acquainted with the list in Vega's tables (1796) going up to 400031, I extended my computation further, confirming that estimate. In 1811, the appearance of Chernau's cribrum gave me much pleasure and I have frequently (since I lack the patience for a continuous count) spent an idle quarter of an hour to count another chiliad here and there; although I eventually gave it up without quite getting through a million. Only some time later did I make use of the diligence of Goldschmidt to fill some of the remaining gaps in the first million and to continue the computation according to Burkhardt's tables. Thus (for many years now) the first three million have been counted and checked against the integral. A small excerpt follows:

Table A

Below	Here are Prime	Integral $\int \frac{dn}{\log n}$	Error	Your Formula	Error
500000	41556	41606.4	+ 50.4	41596.9	+ 40.9
1000000	78501	79627.5	+ 126.5	78672.7	+ 171.7
1500000	114112	114263.1	+ 151.1	114374.0	+ 264.0
2000000	148883	149054.8	+ 171.8	149233.0	+ 350.0
2500000	183016	183245.0	+ 229.0	183495.1	+ 479.1
3000000	216745	216970.6	+ 225.6	217308.5	+ 563.5

I was not aware that Legendre had also worked on this subject; your letter caused me to look in his *Théorie des Nombres*, and in the second edition I found a few pages on the subject which I must have previously overlooked (or, by now, forgotten). Legendre used the formula

$$\frac{n}{\log n - A},$$

where A is a constant which he sets equal to 1.08366. After a hasty computation, I find in the above cases the deviations

TABLE B

— 23,3
+ 42,2
+ 68,1
+ 92,8
+159,1
+167,6

These differences are even smaller than those from the integral, but they seem to grow faster with n so that it is quite possible they may surpass them. To make the count and the formula agree, one would have to use, respectively, instead of $A = 1.08366$, the following numbers:

TABLE C

1,09040
1,07682
1,07582
1,07529
1,07179
1,07297

It appears that, with increasing n, the (average) value of A decreases; however, I dare not conjecture whether the limit as n approaches infinity is 1 or a number different from 1. I cannot say that there is any justification for expecting a very simple limiting value; on the other hand, the excess of A over 1 might well be a quantity of the order of $1/\log n$. I would be inclined to believe that the differential of the function must be simpler than the function itself.

If $dn/\log n$ is postulated for he function, Legendre's formula would suggest that the differential function might be something of the form $dn/(\log n-(A-1))$. By the way, for large n, your formula could be considered to coincide with

$$\frac{n}{\log n-(1/2k)},$$

where k is the modulus of Brigg's logarithms; that is, with Legendre's formula, if we put $A = 1/2k = 1.1513$.

Finally, I want to remark that I noticed a couple of disagreements between your counts and mine.

Between	59000 and	60000, you have	95, while I have	94
	101000	102000	94	93.

The first difference possibly results from the fact that, in Lambert's Supplement, the prime 59023 occurs twice. The chiliad from 101000 — 102000 in Lambert's Supplement is virtually crawling with errors; in my copy, I have indicated seven numbers which are not primes at all, and supplied two missing ones. Would it not be possible to induce young Mr. Dase to count the primes in the following (few) millions, using the tables at the Academy which, I am afraid, are not intended for public distribution? In this case, let me remark that in the 2nd and 3rd million, the count is, according to my instructions, based on a special scheme which I myself have employed in counting the first million. The counts for each 100000 are indicated on a single page in 10 columns, each column belonging to one myriad (*an interval of length* 10000; *Trans.*); an additional column in front (left) and another column following it on the right; for example here is a vertical column and the two additional columns for the interval 10000000 to 11000000 ———

As an illustration, take the first vertical column. In the myriad 1000000 to 1010000 there are 100 Hecatontades; (*intervals of length* 100; *Trans.*) among them one containing a single prime, none containing two or three primes; two containing four each; eleven containing 5 each, etc., yielding altogether $752 = 1.1 + 4.2 + 5.11 + 6.14 + \cdots$ primes. The last column contains the totals from the other ten. The numbers 14, 15, 16 in the first vertical column are superfluous since no hecatontades occur containing that many primes; but on the following pages they are needed. Finally the 10 pages are again combined into one and thus comprise the entire second million.

It is high time to quit — — —. With most cordial wishes for your good health

Yours, as ever,
C. F. Gauss

Göttingen, 24 December 1849.

Appendix C: Corrections to Gauss' Tables

Thousands	Gauss	Actual	Δ
20	102	104	−2
159	87	77	+10
199	96	86	+10
206	85	83	+2
245	78	88	−10
289	85	77	+8
290	84	85	−1
334	80	81	−1
352	80	81	—1
354	79	76	+3
500	Up to here		+18

	Totals		Δ
500,000	41,556	41,538	+18
1,000,000	78,501	78,498*	+3
1,500,000	114,112	114,156*	-44
2,000,000	148,883	148,934*	-51
2,500,000	183,016	183,073*	-57
3,000,000	216,745	216,817*	-72

* from *List of Prime Numbers from* 1 *to* 10,006,771, by D. N. Lehmer, (adjusted: He counts 1 as a prime).

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DIFFERENTIATION UNDER THE INTEGRAL SIGN*

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1. Introduction. Everyone knows the Leibniz rule for differentiating an integral:

$$\frac{d}{dt}\left(\int_{g(t)}^{h(t)} F(x,t)\,dx\right) = \Big\{F[h(t),t]\dot{h}(t) - F[g(t),t]\dot{g}(t)\Big\} + \int_{g(t)}^{h(t)} \frac{\partial F(x,t)}{\partial t}\,dx. \tag{1.1}$$

We are all fond of this formula, although it is seldom if ever used in such generality. Usually, either the limits are constants, or the integrand is independent of the time t. Frequent cases are

$$\frac{d}{dt}\int_a^t F(x)\,dx = F(t), \qquad \frac{d}{dt}\int_0^\infty F(x,t)\,dx = \int_0^\infty \frac{\partial F(x,t)}{\partial t}\,dx.$$

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